Test 1 / Numerical Mathematics 1 / May 9th 2023, University of Groningen

Instructions

- You have 2 hours to complete the test. When applicable, people with special facilities have 2h20 minutes in total.
- The exam is "closed book", meaning that you can only make use of the material given to you.
- The grade will be computed as the number of obtained points, plus 1.
- **Write the answers to Exercises 1 and 2 on separate sheets.**

Consider an arbitrary scalar function $g \in C^3([a,b])$.

Exercise 1

(a) [1.5] Prove that a function approximation method defined via a linear polynomial $z(x)$ such that $g'(a) = z'(a)$ and $g(b) = z(b)$ has the form

$$z(x) = g(b) + g'(a)(x-b), \quad x \in [a,b]. \tag{1}$$

(b) [0.5] Assume you want to approximate $g(x) = \sin x$ in the interval $[0, \pi]$. Draw both $g(x)$ and $z(x)$.

(c) [1.5] Assume now that the "measurements" of the function g and its derivative are obtained with an additive perturbation ε and δ, respectively, i.e. $\hat{g} = g + \varepsilon$ and $\hat{g}' = g' + \delta$. Denote the approximation under perturbations $\hat{z}(x)$. Find an upper bound for $z(x) - \hat{z}(x)$ for all $x = [a,b]$, but where neither x nor evaluations of g and its derivative(s) appear in the bound.

(d) [1] Assume you want to approximate $g(x) = \sin x$ in the interval $[0, \pi]$, now using *piecewise polynomial approximations* based on the approximation method derived in Question (a) using two subintervals. Draw both $g(x)$ and the piecewise approximating functions. Do you expect convergence in the approximation if more subintervals are added? Justify using qualitative arguments, for example by drawing the approximation using three subintervals.

Exercise 2

(e) [1.5] Obtain a numerical integration formula based on the global approximating polynomial of Equation (1).

(f) [2] Determine the degree of exactness of that numerical integration method from Question (e).

(g) [1] Derive a composite integration formula from the method in Question (e), assuming you have a grid of nodes $x_0, \ldots, x_n$. Would the composite numerical integration converge to the exact integral if one would drop the term containing the evaluation of the function derivatives? (answer qualitatively)